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Biased Varisolvent Chebyshev Approximation on Subsets

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Approximation of a continuous function f on an interval $[\alpha, \beta]$ and closed subsets Y by a varisolvent family is considered. The uniform norm is "biased" by weighting positive deviations by a bias factor r. The limit as bias factors tend to infinity and domains Y fill out the interval is examined. If the best one-sided approximation on the interval is of maximum degree, a local existence result holds and convergence to the best one-sided approximation on the interval is uniform on $[\alpha, \beta]$.

Let $C[\alpha, \beta]$ be the space of continuous functions on a closed interval $[\alpha, \beta]$. Let X be any closed non-empty subset of $[\alpha, \beta]$. For $g \in C[\alpha, \beta]$ define

$$\|g\|_X = \sup\{|g(x)| : x \in X\}, \qquad \|g\| = \|g\|_{[\alpha, \beta]}.$$

Let F be an approximating function unisolvent of variable degree on $[\alpha, \beta]$ with parameter space P and bounded degree in the sense of Rice [1; 2, 3]. Let r be a positive number (the bias factor) and define

$$\begin{aligned} d_r(y) &= y, & y &< 0 \\ &= ry, & y &\geqslant 0. \end{aligned}$$

The problem of r-biased approximation (defined in [8, p. 224] in terms of generalized weight functions) on X is, given $f \in C[\alpha, \beta]$, to find a parameter $A^* \in P$ for which $\|d_r(f - F(A, \cdot))\|_X$ attains its infimum $\rho_r(f, X)$

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over $A \in P$. $F(A^*, \cdot)$ is called a best approximation to f on X with respect to the r-biased Chebyshev norm. We can also consider $r = \infty$ and define

$$d_\infty(y) = y, \qquad y \leqslant 0$$
$$= \infty, \qquad y > 0.$$

The problem of one-sided approximation from above (also defined in [8, p. 224] in terms of weight functions) on $[\alpha, \beta]$ is to minimize $\|d_\infty(f - F(A, \cdot))\|$ over $A \in P$, which is equivalent to minimizing $\|f - F(A, \cdot)\|$ subject to the constraint $F(A, \cdot) \geqslant f$ on $[\alpha, \beta]$.

We assume that the difficulty of a constant error curve [3, 4] does not occur. Sufficient conditions for global existence in biased and one-sided approximation are given in [6].

THEOREM [5]. *Let F be of degree n at A. $F(A, \cdot) \geqslant f$ is a best one-sided approximation to f on $[\alpha, \beta]$ if and only if there is a set $x_0, ..., x_n$, $\alpha \leqslant x_0 < \cdots < x_n \leqslant \beta$, such that $f - F(A, \cdot)$ takes alternately the value $-\|d_\infty(f - F(A, \cdot))\|$ and 0 on the set. Best one-sided approximation is unique.*

LEMMA 1. *Let $F(A, \cdot)$ be the best one-sided approximation to f from above on $[\alpha, \beta]$ and F be of degree n at A. Let $\{x_0, ..., x_n\}$ be an ordered set of points such that $f - F(A, \cdot)$ is alternately $-\|d_\infty(f - F(A, \cdot))\|$ and 0. Let $\varepsilon > 0$ be given. Then there exists δ, $0 < \delta < \varepsilon$, such that if $|x_i - x_i'| < \delta$, $r > 2/\delta$, and*

$$\max\{|d_r(f(x_i') - F(B, x_i'))| : i = 0, 1, ..., n\} \leqslant \|d_\infty(f - F(A, \cdot))\|, \qquad (*)$$

then

$$F(B, x_i') - F(A, x_i')$$
$$\geqslant -\varepsilon \|d_\infty(f - F(A, \cdot))\| \qquad \text{if } f(x_i) - F(A, x_i) = 0 \qquad (1)$$
$$\leqslant \varepsilon \|d_\infty(f - F(A, \cdot))\| \qquad \text{if } f(x_i) - F(A, x_i) = -\|d_\infty(f - F(A, \cdot))\|. \qquad (2)$$

Proof. $f - F(A, \cdot)$ is continuous on $[\alpha, \beta]$, hence continuous uniformly on $[\alpha, \beta]$. There exists $\delta_1 > 0$ such that if $|x - y| < \delta_1$,

$$|(f(x) - F(A, x)) - (f(y) - F(A, y))| < \frac{\varepsilon}{2} \|d_\infty(f - F(A, \cdot))\|.$$

Select δ, $0 < \delta < \delta_1$, such that $|x_i - x_i'| < \delta$ implies

$$|(f(x_i) - F(A, x_i)) - (f(x_i') - F(A, x_i'))| < \frac{\varepsilon}{2} \|d_\infty(f - F(A, \cdot))\|. \qquad (3)$$

Supposing that (1) does not hold, we have

$$F(A, x_i') - F(B, x_i') > \varepsilon \, \|d_\infty(f - F(A, \cdot))\| \tag{4}$$

and $f(x_i) - F(A, x_i) = 0$. Then from (3)

$$|f(x_i') - F(A, x_i')| < \frac{\varepsilon}{2} \, \|d_\infty(f - F(A, \cdot))\|. \tag{5}$$

From (4) and (5) we have

$$\begin{aligned} f(x_i') - F(B, x_i') &= f(x_i') - F(A, x_i') + F(A, x_i') - F(B, x_i') \\ &> \frac{\varepsilon}{2} \, \|d_\infty(f - F(A, \cdot))\| > 0, \end{aligned}$$

hence

$$\begin{aligned} d_r(f(x_i') - F(B, x_i')) &= r(f(x_i') - F(B, x_i')) > r \cdot \frac{\varepsilon}{2} \, \|d_\infty(f - F(A, \cdot))\| \\ &> \frac{2}{\delta} \cdot \frac{\varepsilon}{2} \, \|d_\infty(f - F(A, \cdot))\| > \|d_\infty(f - F(A, \cdot))\|. \end{aligned}$$

This contradicts (*).

Supposing that (2) does not hold, we have

$$F(A, x_i') - F(B, x_i') < -\varepsilon \, \|d_\infty(f - F(A, \cdot))\| \tag{6}$$

and

$$f(x_i) - F(A, x_i) = -\|d_\infty(f - F(A, \cdot))\|.$$

Then from (3)

$$\begin{aligned} f(x_i') - F(A, x_i') &< f(x_i) - F(A, x_i) + \varepsilon \, \|d_\infty(f - F(A, \cdot))\| \\ &= (-1 + \varepsilon) \, \|d_\infty(f - F(A, \cdot))\|. \end{aligned} \tag{7}$$

From (6) and (7) we have

$$\begin{aligned} f(x_i') - F(B, x_i') &= f(x_i') - F(A, x_i') + F(A, x_i') - F(B, x_i') \\ &< -\|d_\infty(f - F(A, \cdot))\| < 0, \end{aligned}$$

hence

$$d_r(f(x_i') - F(B, x_i')) = f(x_i') - F(B, x_i') < -\|d_\infty(f - F(A, \cdot))\|,$$

again contradicting (*).

LEMMA 2 [7]. *Let F be of degree n (maximal) at A, then for given* $\varepsilon > 0$ *there exists* $\eta(\varepsilon)$ *such that* $\|F(A, \cdot) - F(B, \cdot)\| < \eta(\varepsilon)$ *if* (1) *holds and* $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

LEMMA 3 [7]. *Let F be unisolvent of degree m at* A_k, $k = 0, 1, ...,$ *and let* $\{F(A_k, \cdot)\}$ *converge pointwise to* $F(A_0, \cdot)$ *on m distinct points. Then* $\{F(A_k, \cdot)\}$ *converges uniformly to* $F(A_0, \cdot)$ *on* $[\alpha, \beta]$.

Define the density of a subset X of $[\alpha, \beta]$ to be

$$|X| = \sup\{\inf\{|x - y| : y \in X\} : \alpha \leqslant x \leqslant \beta\}.$$

We say $\{X_k\} \to [\alpha, \beta]$ if $X_k \subset [\alpha, \beta]$ and for $x \in [\alpha, \beta]$, there is a sequence $\{x_k\} \to x$, $x_k \in X_k$.

THEOREM. *Let F be unisolvent of variable degree. Let f have a best one-sided approximation* $F(A, \cdot)$ *from above on* $[\alpha, \beta]$ *and let F be of degree n (maximal) at A. There exist* $\delta > 0$ *and M such that the density of X being less than* δ *and* $r > M$ *imply that there is a best approximation to f on X with respect to the r-biased Chebyshev norm. Let* $\{X_k\} \to [\alpha, \beta]$ *and* $r(k) \uparrow \infty$ *and* $F(A_k, \cdot)$ *be best approximations on* X_k *with respect to the* $r(k)$*-biased Chebyshev norm. Then* $\{F(A_k, \cdot)\}$ *converges uniformly to* $F(A, \cdot)$ *on* $[\alpha, \beta]$.

Proof. Let $x_0, ..., x_n$ be as in Lemma 1. By definition of solvency of degree n at A there exists $\lambda > 0$ such that if $|y_j - F(A, x_j)| < \lambda$, $j = 1, ..., n$, then there exists a parameter B satisfying

$$F(B, x_j) = y_j, \qquad j = 1, ..., n. \tag{8}$$

Using property Z and maximality of n, it is easily seen that F is unisolvent of degree n at such B, and hence B is completely determined by (8). Choose ε such that $\eta(\varepsilon) < \lambda/2$, then by Lemmas 1 and 2, there exist δ, $0 < \delta < \varepsilon$, such that if $r > 2/\delta$, $|x_i - x_i'| < \delta$, and

$$\max\{|d_r(f(x_i') - F(B, x_i'))| : i = 0, 1, ..., n\} \leqslant \|d_\infty(f - F(A, \cdot))\|,$$

then

$$\|F(A, \cdot) - F(B, \cdot)\| < \lambda/2.$$

Now let the density of X be less than δ and let $\|d_r(f - F(B_k, \cdot))\|_X$ be a decreasing sequence with limit $\rho_r(f, X)$. As $X \subset [\alpha, \beta]$ and from Lemma 4 of [5], $\rho_r(f, X) \leqslant \rho_r(f, [\alpha, \beta]) < \|d_\infty(f - F(A, \cdot))\|$. Let $x_i' \in X$, $|x_i' - x_i| < \delta$, $i = 0, 1, ..., n$, and $r > 2/\delta$. For all k sufficiently large,

$$\max\{|d_r(f(x_i') - F(B_k, x_i'))| : i = 0, 1, ..., n\} \leqslant \|d_\infty(f - F(A, \cdot))\|,$$

hence

$$\|F(A, \cdot) - F(B_k, \cdot)\| < \lambda/2.$$

Then n-tuples of values at the points $x_1, ..., x_n$ of the approximants $F(B_k, \cdot)$ form a bounded sequence with subsequence converging to an accumulation point $(y_1, ..., y_n)$ which determines a parameter B at which F is unisolvent of degree n. By Lemma 3, $\{F(B_k, \cdot)\}$, taking subsequence if necessary, converges uniformly on $[\alpha, \beta]$ to $F(B, \cdot)$, hence for all $x \in X$, $|d_r(f(x) - F(B, x))| \leqslant \rho_r(f, X)$ and so $F(B, \cdot)$ is a best approximation to f on X with respect to the r-biased Chebyshev norm. The first part of the theorem is proved.

Now let $X_k \to [\alpha, \beta]$, $r(k) \uparrow \infty$, then for k sufficiently large a best approximation $F(A_k, \cdot)$ to f on X_k with respect to the $r(k)$-biased Chebyshev norm exists. From Lemmas 1 and 2 it follows that $\{F(A_k, \cdot)\}$ converges uniformly to $F(A, \cdot)$ on $[\alpha, \beta]$.

The results suggest determining the best one-sided approximation on $[\alpha, \beta]$ as the limit of best $r(k)$-biased approximation on a sequence of finite subsets $X_k \to [\alpha, \beta]$.

If the best one-sided approximation to f on $[\alpha, \beta]$ is not of maximum degree, best biased approximation on subsets need not exist and even if it exists, convergence of best approximations on subsets may not be uniform [5].

Let us also consider the case when the bias factor r tends to zero. Positive deviations are weighted by r and negative deviations weighted by 1. This is equivalent to weighting positive deviations by 1 and negative deviations by $1/r$, which increases both deviations by a factor of $1/r$. We get by similar arguments.

THEOREM. *Let F be unisolvent of variable degree. Let f have a best one-sided approximation $F(A, \cdot)$ from below on $[\alpha, \beta]$ and let F be of degree n (maximal) at A. There exist $\delta > 0$ and ε such that the density of X being less than δ and $r < \varepsilon$ imply that there is a best approximation to f on X with respect to the r-biased Chebyshev norm. Let $\{X_k\} \to [\alpha, \beta]$ and $r(k)$ be a decreasing sequence with limit 0 and let $F(A_k, \cdot)$ be best to f on X_k with respect to the $r(k)$-biased Chebyshev norm. Then $\{F(A_k, \cdot)\}$ converges uniformly to $F(A, \cdot)$ on $[\alpha, \beta]$.*

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